

# Invariant Funnels around Trajectories using Sum-of-Squares Programming. <sup>\*</sup>

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**Abstract:** This paper presents numerical methods for computing regions of finite-time invariance (funnels) around solutions of polynomial differential equations. First, we present a method which exactly certifies sufficient conditions for invariance despite relying on approximate trajectories from numerical integration. Our second method relaxes the constraints of the first by sampling in time. In applications, this can recover almost identical funnels but is much faster to compute. In both cases, funnels are verified using Sum-of-Squares programming to search over a family of time-varying polynomial Lyapunov functions. Initial candidate Lyapunov functions are constructed using the linearization about the trajectory, and associated time-varying Lyapunov and Riccati differential equations. The methods are compared on stabilized trajectories of a six-state model of a satellite.

*Keywords:* Nonlinear systems, Lyapunov methods, Stability domains

## 1. INTRODUCTION

In this paper we propose algorithms for computing a finite-time “funnel” of a dynamical system: a set of initial conditions whose solutions are guaranteed to enter a certain goal region at a particular time. This work is motivated by new algorithms for nonlinear control design wherein the controller is constructed from a tree of finite-time trajectories, each locally stabilized and all leading to a certain goal point (see Tedrake et al. (2010)). Sparseness of this tree is advantageous, and is directly related to the size of the funnel that can be verified for each trajectory in the tree.

The basic method is to search over a class of time-varying Lyapunov functions about the trajectory, and verify a positive-invariance condition via Sum-of-Squares programming. A preliminary algorithm for this verification was suggested in Tedrake et al. (2010). This paper extends that work by replacing a greedy set of nonconvex optimizations by an alternation of convex optimizations (similar to that proposed for equilibria in Jarvis-Wloszek et al. (2005)). It is shown that the verified regions can be made exact even if the trajectory is an approximate (numerical) solution. Furthermore, an alternative time-sampled verification is suggested which recovers almost identical funnels and is substantially faster to compute.

Recently, a great deal of research has been dedicated to region-of-attraction analysis on polynomial vector fields (Papachristodoulou and Prajna (2002); Parrilo (2003); Topcu et al. (2008); Tan and Packard (2008)) and more general non-polynomial vector fields (Papachristodoulou and Prajna (2005); Chesi (2009)) through Sum-of-Squares programming. There is comparatively little written in

the Sum-of-Squares literature addressing time-varying systems. In Julius and Pappas (2009) finite time-invariance around trajectories is explored to provide outer approximations of the set reached from some initial conditions. This is accomplished by using regionally valid Lyapunov certificates to construct barrier functions bounding exponentia, certifying solutions do not enter keep-out sets. By contrast, the algorithm in this paper constructs inner-approximations of the solutions which can reach a goal region through the construction of time-varying Lyapunov functions.

### 1.1 Notation

We denote the set of  $n$ -by- $n$  positive definite matrices by  $\mathbb{S}_+^n$ . For  $P \in \mathbb{S}_+^n$  and  $x \in \mathbb{R}^n$  we use  $\|x\|_P^2$  as short hand for  $x'Px$ . We denote the set of polynomials in  $x \in \mathbb{R}^n$  with real coefficients by  $\mathbb{R}[x]$ . The subset of these polynomials which are Sum-of-Squares (SOS) is denoted  $\Sigma[x]$ , that is:  $p(x) \in \Sigma[x]$  if and only if there exists  $\{g_i(x)\}_{i=1}^k \subset \mathbb{R}[x]$  such that  $p(x) = \sum_{i=1}^k g_i(x)^2$ . We similarly denote polynomials and SOS polynomials in multiple vector valued variables  $(x_1, \dots, x_k) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$  by  $\mathbb{R}[x_1, \dots, x_k]$  and  $\Sigma[x_1, \dots, x_k]$ . Finally, for a set  $A \subset \mathbb{R}^n$ ,  $\text{int}(A)$  refers to its interior.

## 2. PRELIMINARIES

We are given a time-varying nonlinear dynamical system:

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad (1)$$

where  $f : [t_0, t_f] \times \mathbb{R}^n \mapsto \mathbb{R}^n$  is piecewise polynomial in  $t$  and polynomial in  $x$ . Further we are given an bounded “goal region”,  $\mathcal{G} \subset \mathbb{R}^n$ , with non-empty interior. In applications, we will generally be investigating systems (1)

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arising as closed loop systems stabilizing some trajectory. For convenience we make the following definitions.

*Definition 1.* Given the dynamics (1), a set  $\mathcal{F} \subset [t_0, t_f] \times \mathbb{R}^n$  is a *funnel* if for each  $(\tau, x_\tau)$  in  $\mathcal{F}$ , the solution to (1) with  $x(\tau) = x_\tau$  exists on  $[\tau, t_f]$  and for each  $t \in [\tau, t_f]$  we have  $(t, x(t)) \in \mathcal{F}$ .

*Definition 2.* Given the dynamics (1) and the goal region  $\mathcal{G}$ , a set  $\mathcal{F} \subset [t_0, t_f] \times \mathbb{R}^n$  is a *funnel into*  $\mathcal{G}$  if it is a funnel and for any  $x \in \mathbb{R}^n$  we have  $(t_f, x) \in \mathcal{F}$  implying  $x \in \mathcal{G}$ .

We see that a funnel into  $\mathcal{G}$  is an inner-approximation of the set of solutions which flow through  $\mathcal{G}$  at the time  $t_f$ . In this work we are interested in finding the largest possible funnel into  $\mathcal{G}$ , measured, for example, by volume as a subset in  $[t_0, t_f] \times \mathbb{R}^n$ . Our approach uses time-varying Lyapunov functions, exploiting the following Lemma.

*Lemma 3.* Let  $V(t, x)$  be a function  $V : [t_0, t_f] \times \mathbb{R}^n \mapsto [0, \infty)$  piecewise continuously differentiable<sup>1</sup> with respect to  $t$  and continuously differentiable with respect to  $x$ . For each  $t \in [t_0, t_f]$  we define:

$$\Omega_t := \{x \mid V(t, x) \leq 1\},$$

and

$$\partial\Omega_t := \{x \mid V(t, x) = 1\}.$$

If for each  $t \in [t_0, t_f]$  and  $x \in \partial\Omega_t$  we have:

$$\frac{\partial}{\partial x} V(t, x) f(t, x) + \frac{\partial}{\partial t} V(t, x) < 0, \quad (2)$$

then the set:

$$\mathcal{F} = \{(t, x) \mid x \in \Omega_t\} \quad (3)$$

is a funnel. If additionally  $\Omega_{t_f} \subset \mathcal{G}$ , then  $\mathcal{F}$  is a funnel into  $\mathcal{G}$ .

To leverage the SOS relaxation, our goal region  $\mathcal{G}$  will be the closed interior of an ellipse:

$$\mathcal{G} = \{x \mid \|x\|_{P_G}^2 \leq 1\},$$

where  $P_G \in \mathbb{R}^{n \times n}$  is a symmetric positive definite matrix. More general goal regions can be defined as sub-level sets of polynomials.

## 2.1 A Class of Lyapunov Functions

We begin by describing a class of candidate Lyapunov functions based on solutions of (1) and related Lyapunov differential equations. To leverage SOS we approximate these solutions by piecewise polynomials in time. This approximation does not render our certificates inexact. Further, we describe how sufficiently fine approximation of these solutions guarantees the existence of a certificate.

We assume we have access to a nominal solution of (1),  $x_0 : [\tau, t_f] \mapsto \mathbb{R}^n$  with  $\tau \in [t_0, t_f]$ , such that  $x_f := x_0(t_f) \in \text{int}(\mathcal{G})$ . We solve for a symmetric positive definite matrix  $P_f \in \mathbb{S}_+^n$  describing the largest volume ellipse centered on  $x_f$ ,

$$\mathcal{E}_f = \{x \mid \|x - x_f\|_{P_f}^2 \leq 1\},$$

contained in the goal region (i.e.  $\mathcal{E}_f \subset \mathcal{G}$ ). When  $\mathcal{G}$  is an ellipse, this containment problem is an SDP. Given a more general polynomial goal region, one can relax the problem

<sup>1</sup> We will generally write conditions on  $\frac{\partial}{\partial t} V$ . At points of discontinuity, we require these conditions to hold for both the left and right derivative of  $V$  with respect to  $t$ .

to finding the largest sphere centered on  $x_f$  contained in  $\mathcal{G}$  to a SOS program.

Our Lyapunov functions are parameterized by a time-varying, symmetric positive definite matrix  $P^* : [\tau, t_f] \mapsto \mathbb{S}_+^n$ :

$$V^*(t, x) = \|x - x_0(t)\|_{P^*(t)}^2. \quad (4)$$

We require  $P^*(t_f) \geq P_f$ , so that  $\Omega_{t_f}$ , the one sub-level set of  $V^*(t_f, x)$ , is a subset of  $\mathcal{E}_f$  and thus contained in the goal region.

Our optimization approach requires an initial feasible candidate Lyapunov function which is then improved via an iterative process. To construct this initial candidate we look at the dynamics linearized about the trajectory by constructing  $A : [\tau, t_f] \mapsto \mathbb{R}^{n \times n}$ , with  $A(t) = \frac{\partial f}{\partial x}(t, x_0(t))$ . Then, fixing a function  $Q : [\tau, t_f] \mapsto \mathbb{S}_+^n$ , we solve the Lyapunov differential equation:

$$-\dot{P}_0^*(t) = A(t)'P_0^*(t) + P_0^*(t)A(t) + Q(t), \quad P_0^*(t_f) = P_f \quad (5)$$

over the interval  $[\tau, t_f]$ . The following Lemma suggests a procedure for finding a candidate Lyapunov function.

*Lemma 4.* Given a solution  $P_0^*(t)$  to the Lyapunov equation above, there exists a positive constant  $c$  such that  $V^*$  defined in equation (4) with:

$$P^*(t) = \exp\left(c \frac{t_f - t}{t_f - \tau}\right) P_0^*(t)$$

satisfies the conditions of Lemma (3), so that:

$$\mathcal{F} = \{(t, x) \in [\tau, t_f] \times \mathbb{R}^n \mid V(t, x) \leq 1\}$$

is a positive funnel into  $\mathcal{G}$ .

**Proof.** We begin by changing coordinates to  $\bar{x}(t) = x(t) - x_0(t)$ , and defining  $\bar{f}(t, \bar{x})$  by:

$$\dot{\bar{x}} = \bar{f}(t, \bar{x}) = f(t, x(t)) - f(t, x_0(t))$$

We can decompose  $\bar{f}(t, \bar{x})$  as:

$$\dot{\bar{x}} = A(t)\bar{x} + \tilde{f}(t, \bar{x})$$

where  $\tilde{f}(t, \bar{x})$  consists of the second and higher-order terms in  $\bar{f}(t, \bar{x})$ . Taking

$$V(t, x) = \exp\left(c \frac{t_f - t}{t_f - \tau}\right) \bar{x}(t)' P_0^* \bar{x}(t)$$

we have

$$\begin{aligned} \dot{V}(t, x) &= \exp\left(c \frac{t_f - t}{t_f - \tau}\right) [2\bar{x}' P_0^* \bar{f}(t, \bar{x}) + \bar{x}' (\dot{P}_0^* - c P_0^*) \bar{x}] \\ &= \exp\left(c \frac{t_f - t}{t_f - \tau}\right) [2\bar{x}' P_0^* \tilde{f}(t, \bar{x}) - \bar{x}' (Q + c P_0^*) \bar{x}]. \end{aligned}$$

Now,  $\partial\Omega_{t_f}$  is a compact set, so  $\tilde{f}(t_f, x)$  is bounded for  $x \in \partial\Omega_{t_f}$ , therefore there exists a sufficiently large  $c = c_1$  such that  $\dot{V}(t_f, x) < 0$  for  $x \in \partial\Omega_{t_f}$ .

Since  $P(t) > 0$  and is continuous in  $t$  and  $\dot{V}$  is continuous in  $x$  and  $t$ , there exists a time  $t_m < t_f$  such that  $\dot{V}(t, x) < 0$  for all  $x \in \partial\Omega_t$  for all  $t \in [t_m, t_f]$ .

We now show that this is also true on  $t \in [\tau, t_f]$ . Since  $t_m < t_f$ , for any  $\epsilon > 0$  there exists a  $c$  sufficiently large that the sets  $\partial\Omega_t$  are contained in the ball  $|\bar{x}| < \epsilon$  for all  $t \in [\tau, t_m]$ .

Since  $\bar{x}'P_0^*\tilde{f}(t, \bar{x})$  contains third and higher orders in  $\bar{x}$ , there exists a sufficiently small  $\epsilon$  such that  $|\bar{x}(t)'P_0^*(t)\tilde{f}(x, t)| < \bar{x}(t)'(Q(t) + cP_0^*(t))\bar{x}(t)$  for all  $\bar{x} \in \partial\Omega_t$  for all  $t \in [\tau, t_m]$ .

This implies that there exists a sufficiently large  $c = c_2$  such that  $\bar{V}(t, x) < 0$  for all  $x \in \partial\Omega_t$  for all  $t \in [\tau, t_m]$ .

Taking  $c = \max\{c_1, c_2\}$  proves the Lemma.  $\square$

As we only consider a finite time interval, this result guarantees the funnel will have a non-empty intersection with  $\{t\} \times \mathbb{R}^n$  for each  $t \in [\tau, t_f]$ .

## 2.2 Polynomial Lyapunov Functions

To exploit SOS programming, we develop an alternative Lyapunov function to (4) defined by a piecewise polynomial functions:  $\hat{x}_0 : [\tau, t_f] \mapsto \mathbb{R}^n$ ,  $P_0 : [\tau, t_f] \mapsto \mathbb{S}_+^n$  and  $\rho : [\tau, t_f] \mapsto (0, \infty)$ :

$$V(t, x) = \|x - \hat{x}_0(t)\|_{P(t)}^2, \quad P(t) = \frac{P_0(t)}{\rho(t)}. \quad (6)$$

In particular, we have  $\hat{x}_0(t)$  approximate  $x_0(t)$  and  $P_0(t)$  approximate  $P_0^*(t)$ . The function  $\rho(t)$  is a time-varying rescaling which we describe in the next section.

Note that the approximate nature of  $\hat{x}_0(t)$  does not preclude  $V(t, x)$  from verifying a funnel as the conditions in Lemma 3 are concerned with the *level-set* where  $V(t, x) = 1$ . As a result, the exact behavior of the function near  $V(t, x) = 0$  is not essential.

## 3. OPTIMIZATION PROCEDURE

Once we restrict ourselves to a class of piecewise polynomial Lyapunov functions, we can approach our optimization task as a bilinear Sum-of-Squares program. In particular, we will see that the conditions of Lemma (2) will be tests of polynomial negativity on semi-algebraic sets. To verify conditions on these specific time-intervals and level-sets, we make use of the polynomial  $\mathcal{S}$ -procedure (see Parrilo (2003)). We arrive in a program with constraints bilinear in our parameterization of  $V(t, x)$  and multipliers used in the  $\mathcal{S}$ -procedure. In particular, for this work we parameterize  $V$  solely by our choice of time-varying  $\rho(t)$ .

Let  $\{t_i\}_{i=0}^N$  be a set of knot points which contains the knot points of  $f(t, x)$ ,  $\hat{x}_0(t)$ , and  $P_0(t)$ . In particular, we order the knots so that  $t_i < t_{i+1}$  for  $i \in \mathcal{N} = \{0, \dots, N-1\}$ .

We define:

$$\bar{V}(t, x) = \|x - \hat{x}_0(t)\|_{P_0(t)}^2 \quad (7)$$

For each interval  $[t_i, t_{i+1}]$  we define Lagrange multipliers  $\ell_i, \mu_i \in \mathbb{R}[t, x]$ . Let  $\rho_i, f_i$  and  $\bar{V}_i$  be the polynomial pieces of  $\rho, f$  and  $\bar{V}$  on the these intervals. For some constant  $\varepsilon > 0$ , we will attempt to optimize a cost function  $h(\rho)$  according to:

$$\begin{aligned} \max_{\{\rho_i, \ell_i, \mu_i\}_{i \in \mathcal{N}}} \quad & h(\rho) \\ \text{subj. to} \quad & \rho_{N-1}(t_f) = 1, \\ & \forall i \in \mathcal{N} : \\ & \rho_i(t) \in \Sigma[t], \rho_i(t_{i+1}) = \rho_{i+1}(t_{i+1}), \\ & \varepsilon - \left[ \frac{\partial}{\partial x} \bar{V}_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} \bar{V}_i(t, x) \right. \\ & \quad \left. - \dot{\rho}_i(t) + \mu_i(t, x)(\rho_i(t) - \bar{V}_i(t, x)) \right. \\ & \quad \left. + \ell_i(t, x)(t - t_i)(t_{i+1} - t) \right] \in \Sigma[t, x], \\ & \ell_i \in \Sigma[t, x]. \end{aligned} \quad (8)$$

We note that the volume of the set  $\mathcal{F}$  defined by  $V(t, x)$  is proportional to:

$$\text{vol}(\mathcal{F}) \propto \int_{t_0}^{t_f} \sqrt{\frac{\rho(t)^n}{\det(P_0(t))}} dt.$$

As a surrogate for this cost function, we optimize the linear cost:

$$h(\rho) = \int_{t_0}^{t_f} \rho(t) dt$$

which can be computed exactly.

We demonstrate that if a given set  $\{\rho_i, \ell_i, \mu_i\}_{i \in \mathcal{N}}$  is feasible, then  $\rho(t)$  defines a function  $V(t, x)$  satisfying the conditions of Lemma 2. First, we see that  $\rho$  is constrained to be continuous and positive so that  $V(t, x)$  will be piecewise continuous, piecewise continuously differentiable and positive.

Writing  $V_i = \frac{\bar{V}_i}{\rho_i}$ , we have:

$$\frac{\partial}{\partial x} V_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} V_i(t, x) \quad (9)$$

equivalent to:

$$\frac{1}{\rho_i(t)} \left[ \frac{\partial}{\partial x} \bar{V}_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} \bar{V}_i(t, x) - \dot{\rho}_i(t) \frac{\bar{V}_i(t, x)}{\rho_i(t)} \right].$$

As a result, for  $t \in [t_i, t_{i+1}]$  and  $x$  such that  $\bar{V}_i(t, x) = \rho_i(t)$  (equivalently,  $x \in \partial\Omega_t$ ) the optimization (8) verifies:

$$\frac{\partial}{\partial x} V_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} V_i(t, x) < \varepsilon \rho(t)^{-1}.$$

Note that these constraints verify that:

$$\frac{\partial}{\partial x} V(t, x) f(t, x) + \frac{\partial}{\partial t} V(t, x) < 0$$

where we impose the constraint on both left and right derivatives with respect to time where  $\frac{\partial}{\partial t}$  is not continuous.

This optimization is unfortunately bilinear in the coefficients of  $\rho$  and the Lagrange multipliers. However, this is amenable to an alternating search. Given a feasible  $\rho(t)$ , we can compute Lagrange multipliers. Then, holding these multipliers fixed we can attempt to improve  $\rho(t)$ . This is described in detail below.

### 3.1 The Multiplier-Step: Finding LaGrange Multipliers

For a fixed  $\rho(t)$  we can compute Lagrange multipliers via the following set of optimizations. For each interval  $[t_i, t_{i+1}]$  we optimize over two polynomials,  $\mu_i$  and  $\ell_i$  in  $\mathbb{R}[t, x]$ , and a slack variable  $\gamma_i$ :

$$\begin{aligned}
& \min_{\gamma_i, \ell_i, \mu_i} \quad \gamma_i \\
& \text{subj. to} \quad \gamma_i - \left[ \frac{\partial}{\partial x} \bar{V}_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} \bar{V}_i(t, x) \right. \\
& \quad \left. - \dot{\rho}_i(t) + \mu_i(t, x)(\rho_i(t) - \bar{V}_i(t, x)) \right. \\
& \quad \left. + \ell_i(t, x)(t - t_i)(t_{i+1} - t) \right] \in \Sigma[t, x], \\
& \quad \ell_i(t, x) \in \Sigma[t, x].
\end{aligned} \tag{10}$$

These programs can be computed in parallel. If each  $\gamma_i$  is negative, then the set  $\{\mu_i, \ell_i, \rho_i\}_{i \in \mathcal{N}}$  is a feasible point for the optimization (8).

An obvious question is how to first obtain a feasible  $\rho(t)$ . Motivated by Section 2, we suggest the following search. We search over a positive constant  $c \geq 0$ , and take  $\rho(t)$  to be a continuous piecewise polynomial approximation:

$$\rho(t) \approx \exp \left( -c \frac{t_f - t}{t_f - \tau} \right).$$

If a given choice of  $c$  does not verify a funnel (i.e. if the optimal values of (10) are not all negative), we iteratively increase the value of  $c$ , and potentially the number of knot point of  $\rho(t)$ .

### 3.2 The V-Step: Improving $\rho(t)$ .

Having solved (10) for each time interval, we now attempt to increase the size of the region verified by  $V$  by optimizing to increase  $\rho(t)$ . To do so, we pose the following optimization with  $\{\ell_i, \mu_i\}_{i \in \mathcal{N}}$  fixed from a solutions to (10):

$$\begin{aligned}
& \max_{\{\rho_i\}_{i \in \mathcal{N}}} \quad h(\rho) \\
& \text{subj. to} \quad \rho_{N-1}(t_f) = 1, \\
& \quad \forall i \in \mathcal{N} : \\
& \quad \rho_i(t) \in \Sigma[t], \rho_i(t_{i+1}) = \rho_{i+1}(t_{i+1}), \\
& \quad \varepsilon - \left[ \frac{\partial}{\partial x} \bar{V}_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} \bar{V}_i(t, x) \right. \\
& \quad \left. - \dot{\rho}_i(t) + \mu_i(t, x)(\rho_i(t) - \bar{V}_i(t, x)) \right. \\
& \quad \left. + \ell_i(t, x)(t - t_i)(t_{i+1} - t) \right] \in \Sigma[t, x].
\end{aligned} \tag{11}$$

So long as the class of  $\rho_i(t)$  includes the  $\rho_i(t)$  used in the optimizations (10), this optimization will be feasible, and can only improve the achieved value of  $h(\rho)$ .

## 4. SAMPLING IN TIME

In terms of computational complexity, the most immediate limiting factor to the above approach is performing Sum-of-Squares optimization is the degree of  $t$  in the polynomials being constrained. We now discuss an approximation to verifying the funnels based on sampling in time.

The proposed approximation validates the conditions of Lemma 3 only at finely sampled points. For each interval  $[t_i, t_{i+1}]$  from the above formulation, we choose a finer sampling  $t_i = \tau_{i1} < \tau_{i2} < \dots < \tau_{iM_i} = t_{i+1}$ . For each such  $\tau_{ij}$ , we define a Lagrange multiplier  $\mu_{ij} \in \mathbb{R}[x]$ . We pose the bilinear SOS optimization:

$$\begin{aligned}
& \max_{\{\rho_i, \mu_{ij}\}_{i \in \mathcal{N}}} \quad h(\rho) \\
& \text{subj. to} \quad \rho_{N-1}(t_f) = 1, \\
& \quad \forall i \in \mathcal{N}, \forall j \in \{1, \dots, M_i\} : \\
& \quad \rho_i(t) \in \Sigma[t], \rho_i(t_{i+1}) = \rho_{i+1}(t_{i+1}), \\
& \quad \varepsilon - \left[ \frac{\partial}{\partial x} \bar{V}_i(\tau_{ij}, x) f_i(\tau_{ij}, x) \right. \\
& \quad \left. + \frac{\partial}{\partial t} \bar{V}_i(\tau_{ij}, x) - \dot{\rho}_i(\tau_{ij}) \right. \\
& \quad \left. + \mu_{ij}(x)(\rho_i(\tau_{ij}) - \bar{V}_i(\tau_{ij}, x)) \right] \in \Sigma[x].
\end{aligned} \tag{12}$$

This optimization removes all algebraic dependence on  $t$ , however it does not in general provide an exact certificate. One would hope that with sufficiently fine sampling one will recover exactness. A partial result to this effect exists.

*Lemma 5.* Let  $V_i : [t_i, t_{i+1}] \times \mathbb{R}^n$  and  $f_i : [t_i, t_{i+1}] \times \mathbb{R}^n \mapsto \mathbb{R}^n$  be continuously differentiable functions of  $t, x$ . Further, say that for all  $t \in [t_i, t_{i+1}]$  and  $x$  such that  $V_i(t, x) = 1$ ,  $\frac{\partial}{\partial x} V(t, x) \neq 0$ .

Then, if there exists a  $t \in [t_i, t_{i+1}]$  and  $x \in \mathbb{R}^n$  such that  $V(t, x) = 1$  and:

$$g(t, x) := \frac{\partial}{\partial x} V_i(t, x) f_i(t, x) + \frac{\partial}{\partial t} V_i(t, x) > \delta$$

for some positive  $\delta$ , then there exists an open interval around  $t$  such that for  $\tau$  in that interval there exists a  $y \in \mathbb{R}^n$  with:

$$V(\tau, y) = 1, \text{ and } g(\tau, y) > 0.$$

**Proof.** As  $g$  is continuous in  $x$  there exists an  $\eta > 0$  such that for all  $z$  in  $B(x, \eta) = \{z \mid \|x - z\| < \eta\}$  we have  $g(t, z) > \delta/2$ . As  $\frac{\partial}{\partial x} V_i(t, x) \neq 0$ , there exists a  $z_1, z_2 \in B(x, \eta)$  such that  $V_i(t, z_1) = 1 + \varepsilon_1$  and  $V_i(t, z_2) = 1 - \varepsilon_2$  for some constants  $\varepsilon_1, \varepsilon_2 > 0$ . As  $\frac{\partial}{\partial t} V_i$  is continuous, and  $B(x, \eta)$  is bounded,  $\frac{\partial}{\partial t} V_i$  is bounded. So there exists an interval around  $t$  such that, for any  $\tau$  in this interval,  $V_i(\tau, z_1) > 1, V_i(\tau, z_2) < 1$ . As  $g$  is continuous in  $t$  there exists a sub-interval such that for  $\tau$  in this interval additionally  $g(\tau, z) > 0$  for all  $z \in B(x, \eta)$ . As  $V_i(t, x)$  is continuous, there exists a  $\theta \in (0, 1)$  such that for  $y = \theta z_1 + (1 - \theta) z_2$  we have  $V_i(\tau, y) = 1$ . But, as  $B(x, \eta)$  is convex,  $y \in B(x, \eta)$  so that  $g(\tau, y) > 0$ .

Clearly the function  $V_i$  we have considered thus far satisfy the requirements of this Lemma. The result suggests that given a  $V(t, x)$  that does not satisfy the conditions of Lemma 3, there exists a sufficiently fine uniform sample spacing such that  $V(t, x)$  will not be in the feasible set of (12). This result is partial in that it does not construct a sufficient sampling bound from computable quantities, and further applies only to a fixed  $V(t, x)$ , whereas we optimize over  $V$  with fixed sampling intervals. We are currently studying ways of integrating more constructive bounds into the optimization (12).

We use an analogous strategy of bilinear alternation to approach (12). The same strategy is used to find an initial feasible  $\rho(t)$ , and then Lagrange multipliers and  $\rho(t)$  are iteratively improved.

## 5. NUMERICAL EXPERIMENTS

We first illustrate the general procedure with a one-dimensional polynomial system. Our second example is an idealized three degree of freedom satellite model. For this second example we compare numerically the proposed techniques.

### 5.1 A One-Dimensional Example

We examine a one dimensional time-varying polynomial differential equation:

$$\frac{d}{dt}x(t) = f(t, x(t)) = x - \frac{1}{2}x^2 + 2t - 2.4t^3, \quad (13)$$

over the interval  $t \in [-1, 1]$ . Our goal region is  $\mathcal{G} = [0, 1]$ . As the system has no control input, we can solve nearly exact bounds on the region  $\mathcal{F}^* \subset [-1, 1] \times \mathbb{R}$  which flows into the goal by computing the solutions to (13) with final value conditions  $x(1) = 1$  and  $x(1) = 0$ .

To find our inner approximation  $\mathcal{F} \subset \mathcal{F}^*$ , we compute an approximate numerical trajectory,  $\hat{x}_0(t)$  with final value  $\hat{x}_0(1) = x_f = 0.5$ . We take  $P_f = 4$  so that  $\mathcal{E}_f = \{x \mid |x - x_f|_{P_f}^2 \leq 1\} = [0, 1]$ . We numerically solve the Lyapunov equation (5) to determine  $P(t)$ .

We use  $N = 40$  knot points,  $\{t_i\}_{i=1}^N$ , chosen to be the steps of a the variable time-step integration of the differential equation Lyapunov. We interpolate  $\hat{x}_0(t)$  with a piecewise cubic polynomial and  $P(t)$  with a piecewise linear function. To find our initial candidate Lyapunov function, we begin by taking  $\rho(t)$  to be a piecewise linear interpolation of  $\exp\left(\frac{c(t-1)}{2}\right)$ , for  $c \geq 0$ . Taking  $c = 4$  provides a feasible candidate Lyapunov function. This feasible solution is then improved by bilinear alternation. Both the initial and optimized sets are plotted against the known bounds in Figure 1.

After a single bilinear alternation, a tight region is found. Note that the symmetry of the Lyapunov function around the trajectory restricts the region being verified. Additional trajectories could be used to continue to grow the verified region.

### 5.2 Trajectory Stabilization of Satellite Dynamics

We next evaluate a time-varying positively invariant region around a feedback stabilized nominal trajectory. In past work, Tedrake et al. (2010), it was demonstrated how trajectory optimization and randomized search can be combined with such certificates to approximate the controllable region for a smooth nonlinear system.

We examine the stabilization of a nominal trajectory for a rigid body floating in space subject to commanded torques. The state of the system is given in terms of the angular velocity of the body about its principal axes,  $\omega \in \mathbb{R}^3$ , and the Modified Rodriguez parameters,  $\sigma \in \mathbb{R}^3$ . Any trajectory which excludes full rotations can be represented by this projection of the quaternion representation of orientation. The kinematic equations are given by:

$$\dot{\sigma} = \frac{1}{4} \left( (1 - \|\sigma\|^2)I + 2\sigma\sigma' - 2 \begin{bmatrix} 0 & \sigma_3 & \sigma_2 \\ \sigma_3 & 0 & \sigma_1 \\ \sigma_2 & \sigma_3 & 0 \end{bmatrix} \right) \omega. \quad (14)$$

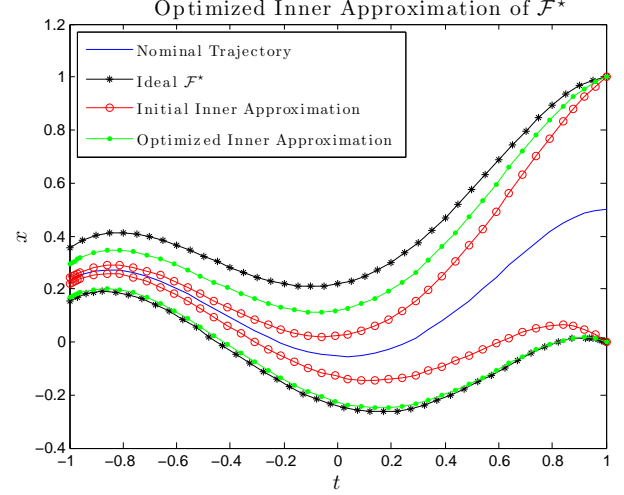


Fig. 1. The ideal set  $\mathcal{F}$  and inner approximations calculated by the method. Surrounding the nominal trajectory (solid blue) are time-varying intervals. An initial candidate Lyapunov function (red open circle) is then improved via the bilinear optimization (solid green circle). In this case, a single step of alternation provided a certificate tight to the known bounds (black stars). Note that the certificate is symmetric about the trajectory, and as a result is generally sub-optimal.

The dynamic equations are given by:

$$H\dot{\omega} = -(\omega \times H\omega) + u \quad (15)$$

where  $H = H' > 0$  is the diagonal, positive definite inertia matrix of the system and  $u \in \mathbb{R}^3$  is a vector of torques. In our example  $H = \text{diag}([5, 3, 2])$ . The state of the system is  $x = [\sigma', \omega']' \in \mathbb{R}^6$ . Together, (14) and (15) define controlled dynamics:

$$\dot{x}(t) = f_0(x(t), u(t)). \quad (16)$$

We design a control policy  $u(t) = \pi(t, x(t))$  such that the closed loop system,

$$\dot{x}(t) = f(t, x(t)) = f_0(x(t), \pi(t, x(t))), \quad (17)$$

satisfies the assumptions of our method.

Our goal region is defined by an ellipse centered on the origin, described by the positive definite matrix:

$$P_G = \begin{bmatrix} 36.1704 & 0 & 0 & 12.1205 & 0 & 0 \\ 0 & 17.4283 & 0 & 0 & 7.2723 & 0 \\ 0 & 0 & 9.8911 & 0 & 0 & 4.8482 \\ 12.1205 & 0 & 0 & 9.1505 & 0 & 0 \\ 0 & 7.2723 & 0 & 0 & 7.3484 & 0 \\ 0 & 0 & 4.8482 & 0 & 0 & 6.2557 \end{bmatrix}.$$

The ellipsoidal region,  $\mathcal{G} = \{x \mid \|x\|_{P_G}^2 \leq 1\}$  was computed numerically as an inner-approximation of the region of attraction for the system after feedback stabilization of the origin.

We begin with a nominal command:

$$u_0(t) = \frac{1}{100}t(t-5)(t+5) \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Table 1. Runtime comparison for exact and sample based approaches.

	Multiplier Step (sec)		V Step (sec)	
Sampled	111	112	220	220
Exact	5316	5357	1336	1420

on the interval  $t \in [0, 5]$ . We compute the solution,  $x_0(t)$  to (16) with  $u(t) \equiv u_0(t)$  and  $x(5) = 0$ . Next, we design a time-varying LQR controller around the trajectory based on the dynamics linearized about the trajectory. Given two cost matrices,  $R \in \mathbb{S}_+^m$  and  $Q \in \mathbb{S}_+^n$ , we solve the Riccati differential equation:

$$-\dot{S}^* = A'S^* + S^*A + Q - S^*BR^{-1}B'S^*, \quad S^*(5) = P_f$$

where  $P_f = 1.01P_G$  and:

$$A(t) = \frac{\partial}{\partial x} f_0(x_0(t), u_0(t)), \quad B(t) = \frac{\partial}{\partial u} f_0(x_0(t), u_0(t)).$$

This procedure gives us a time-varying gain matrix:

$$K(t) = R^{-1}B(t)'S^*(t)$$

The ideal policy is given by:

$$\pi^*(t, x(t)) = u_0(t) - K(t)(x - x_0(t)).$$

To force  $\pi(t, x)$  to be piecewise polynomial in  $t$  and polynomial in  $x$  we take a piecewise constant approximation of  $\hat{K}$  of  $K$  and a piecewise cubic approximation  $\hat{x}_0(t)$  of  $x_0(t)$ . Our control policy is then:

$$\pi(t, x(t)) = u_0(t) - \hat{K}(t)(x - \hat{x}_0(t)).$$

We now examine the closed loop dynamics using the Lyapunov function:

$$V(t, x) = \|x - \hat{x}_0(t)\|_{S_0(t)/\rho(t)}^2$$

where  $S_0(t)$  is a piecewise linear approximation of  $S^*(t)$ . All of the above approximations were piecewise with  $N = 49$  knot points chosen by a variable time-step integration of the Riccati differential equation.

We compare the time required to compute an exact certificate versus the sample based approximation verifying necessary conditions at  $M_i = 10$  points in each interval  $[t_i, t_{i+1}]$ . In the case of computing exact certificates, the expression for  $\frac{d}{dt} V(t, x(t))$  is degree 7 in  $t$ . For both processes, we found that the exponentially decaying initial  $\rho(t)$  with  $c = 3$  was feasible. Figure 2 shows the progress of two iterations using the exact method against the initial  $\rho(t)$ . The figure plots the volume of  $\Omega_t$  for each  $t \in [0, 5]$ . We have found that typically very few iterations are required for the procedure to converge; indeed, in this case the first iteration was already very good. Figure 3 compares the results from two iterations of the time-sampled and exact methods. In this instance they are nearly identical. Table 1 compares the run times. The Multiplier Step consisted of 48 independent SDPs for the exact method and 480 independent SDPs for the sample based method. These computations were performed on a 12-core 3.33 GHz Intel Xeon computer with 24 Gb of RAM. These Multiplier Step programs could be trivially parallelized for speedup.

## 6. DISCUSSION AND FUTURE WORK

We now discuss simple extensions and implementation details for the method.

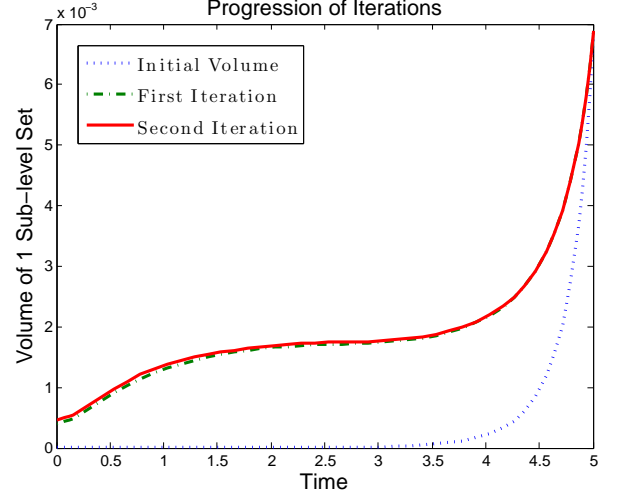


Fig. 2. The result of successive iterations optimizing the volume of verified funnels. After two iterations, the methods, the volume ceases to improve.

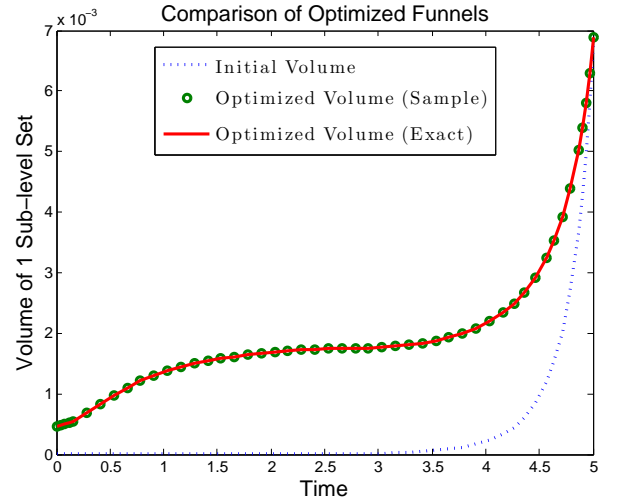


Fig. 3. Comparison of optimized volumes using the exact method and time-sampled method.

### 6.1 Sampling in Time without Splines

When applying the time-sampling method described in this paper, one is no longer bound to finding a polynomial Lyapunov function. As a result, one can use the numerically determined  $x_0(t)$  and  $P_0^*(t)$  instead of interpolations. We make use of the splined quantities in both the sampled based and exact methods in this work only for the sake of comparison.

### 6.2 Verifying Families of Funnels

For several applications, it may prove useful to verify that:

$$\frac{\partial}{\partial x} V(t, x) f(t, x) + \frac{\partial}{\partial t} V(t, x) < 0$$

for all level-sets in a range  $V(t, x) \in [a, 1]$ , with  $0 < a < 1$ . For example: when performing real-time composition of preplanned trajectories, it may not be known in advance what the goal region is for each trajectory segment. By verifying funnels for a whole family of goal regions, the best verified funnel can be chosen at execution time.

### 6.3 More General Lyapunov Functions

In this paper work we have restricted ourselves to rescalings of a time-varying quadratic form, centered near trajectories. Our first numerical example demonstrated how this class can be quite conservative. In principle more general polynomial Lyapunov function can be addressed with the method presented. Using a richer class of polynomial Lyapunov functions has proven advantageous for time-invariant region-of-attraction analysis (for example, see Topcu et al. (2008)). We are currently investigating extending the given algorithm for this case.

### 6.4 Stability of Limit Cycles

The authors have also adapted the procedure proposed in this paper to verify regions of attraction for limit cycles of hybrid systems (see Manchester (2010); Manchester et al. (2010)).

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